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d) $p=1$ - later.

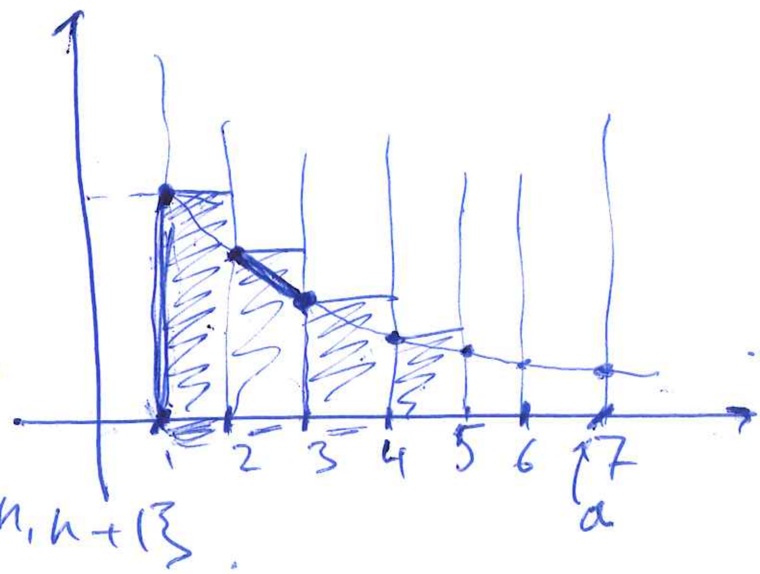
Theorem (integral test). Assume f is continuous, nonnegative, nonincreasing on $[1,\infty)$. Then

$$\int_1^\infty f(x)\,dx \quad \text{and} \quad \sum_{n=1}^{\infty} f(n)$$

converge or diverge together.

$$\cancel{\int_1^\infty f(x)\,dx = \lim_{a\to\infty} \int_1^a f(x)\,dx}$$

Proof.



Suppose $a=n+1$ for some $n\in\mathbb{N}$. Consider the partition

$$P=\{\overset{=x_0}{1},\overset{=x_1}{2},\overset{=x_2}{3},\dots,n-2,n-1,n,n+1\}$$

Then, because f is nonincreasing,

$$\inf_{[x_{i-1},x_i]} f(x) = f(x_i), \quad \sup_{[x_{i-1},x_i]} f(x) = f(x_{i-1}).$$

Therefore,

$$U(f,P) = \sum_{i=1}^{n} f(i),$$

$$L(f,P) = \underbrace{\sum_{i=2}^{n+1} f(i) = \sum_{i=1}^{n} f(i+1)}.$$

Then

$$\underline{L(f,P) \le \int_1^{n+1} f(x)\,dx \le U(f,P)}$$

If $\int_1^{\infty} f(x)\,dx$ converges, then

$L(f,P) = \sum_{i=2}^{n+1} f(i)$ converges as $n \to \infty$,

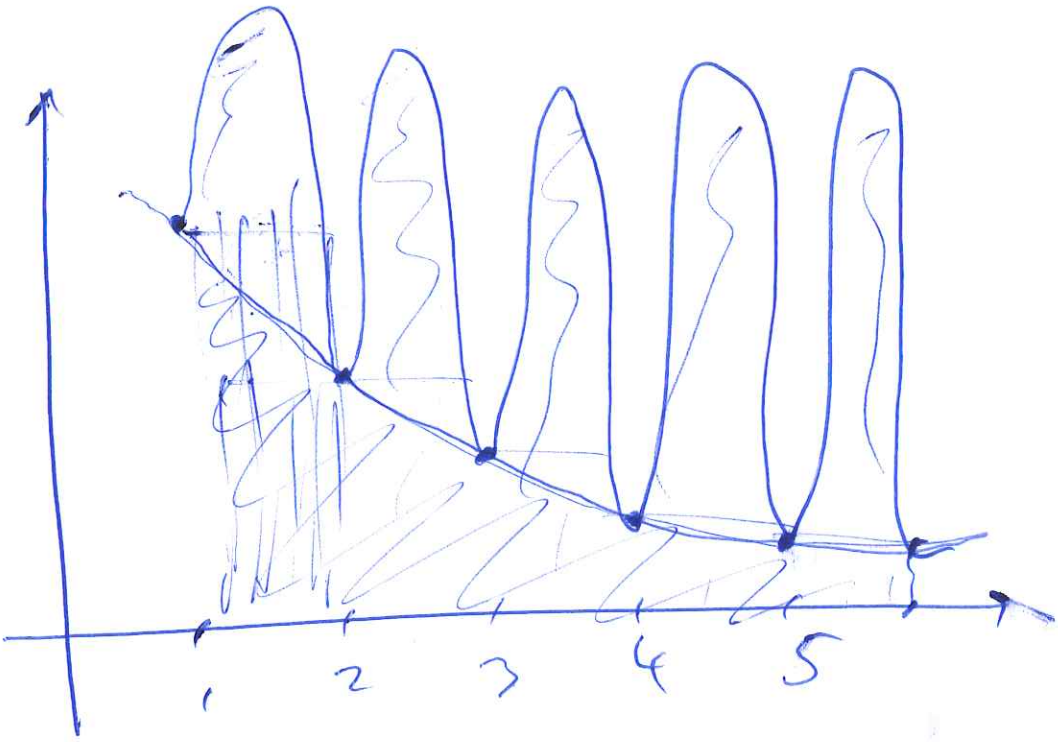
so $\sum_{n=2}^{\infty} f(n)$ converges, so $\sum_{n=1}^{\infty} f(n)$ conv.

If $\int_1^{\infty} f(x)\,dx$ diverges, then

$U(f,P) = \sum_{i=1}^{n} f(i)$ diverges as $n \to \infty$,

so $\sum_{n=1}^{\infty} f(n)$ diverges. $\square$

<u>Remark</u>. Remember that f has to be nonincreasing and continuous.



<u>Side note.</u> Suppose $f: I \to \mathbb{R}$ (I is an open interval) is differentiable and f^{-1} is the inverse of f.

(e.g., $f(x) = x^2$, then $f^{-1}(x) = \sqrt{x}$).

What is $(f^{-1})'$? The chain rule implies:

$f(f^{-1}(x)) = x$. Differentiate both sides:

$f'(f^{-1}(x)) \boxed{(f^{-1})'(x)} = 1$. Therefore,

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

For example,

$f(x) = x^n$ on $[0, \infty)$, n even or
$\mathbb{R}$ if n is odd.

$f^{-1}(x) = x^{\frac{1}{n}}$.

$(f^{-1})'$: If $x \neq 0$, then

$$(x^{\frac{1}{n}})' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{n(x^{\frac{1}{n}})^{n-1}}$$

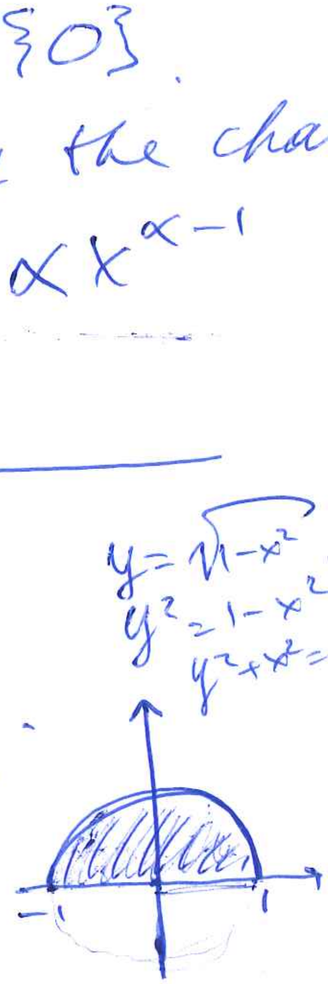
$$= \frac{1}{n} x^{-\frac{1}{n}(n-1)} = \frac{1}{n} x^{\frac{1}{n}-1}$$

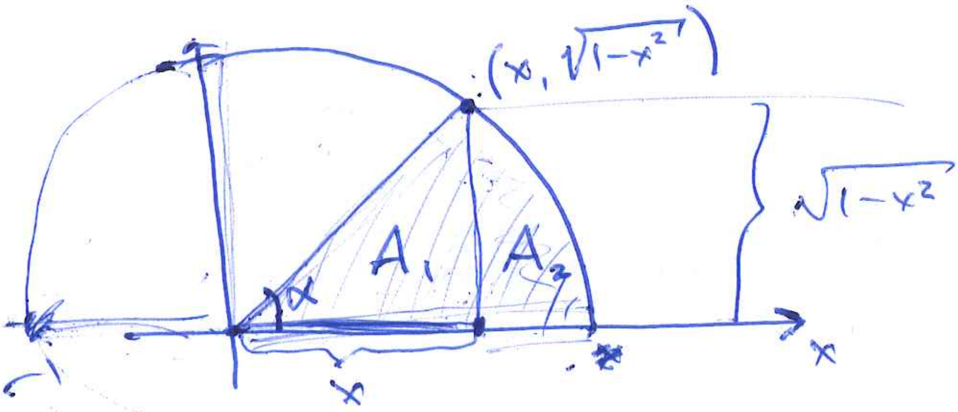
Thus, $(x^{\alpha})' = \alpha x^{\alpha-1}$ for all $\alpha \in \mathbb{Z}$ and all $\frac{1}{n}$, where $n \in \mathbb{Z} \setminus \{0\}$.
We can extend this using the chain rule to conclude $(x^{\alpha})' = \alpha x^{\alpha-1}$ for all $\alpha \in \mathbb{Q}$.

<u>Defining the cos.</u>

$y = \sqrt{1-x^2}$,
$y^2 = 1 - x^2$,
$y^2 + x^2 = 1$.

<u>Definition.</u> $\pi = 2 \int_{-1}^{1} \sqrt{1-x^2}\, dx$.





$A(x) = \frac{\alpha}{2}$.

$A(\cos\alpha) = \frac{\alpha}{2}$.

Define $A(x)$ to be the area of the shaded region, i.e., $A(x) = A_1 + A_2$.

Clearly, $A_1 = \frac{1}{2} x \sqrt{1-x^2}$,

$A_2 = \int_x^1 \sqrt{1-s^2}\, ds$.

Now, $\underline{A(x) = \frac{1}{2} x \sqrt{1-x^2} + \int_x^1 \sqrt{1-s^2}\, ds}$,

Check: $A'(x) = -\frac{1}{2\sqrt{1-x^2}}$.

$A(x)$ is decreasing on $[-1, 1]$ from 0 to $\frac{\pi}{2}$. Thus, $A(-1) = \frac{\pi}{2}$, $A(1) = 0$.

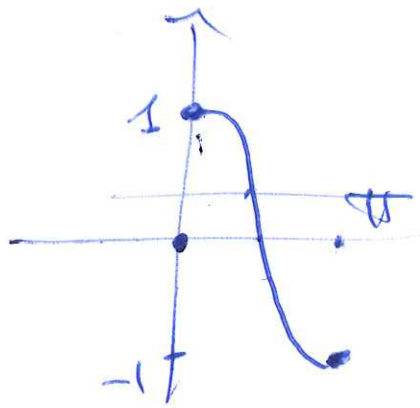
<u>Def.</u> If $x \in [0, \pi]$, then $\cos x$ is defined as the unique number s.t. $A(\cos x) = \frac{x}{2}$.

Why is $\cos x$ well-defined?

It is easy to see that $\cos 0 = 1$ $(A(\cos 0) = 0)$,

$\cos(\pi) = -1 \quad (A(\cos\pi) = \frac{\pi}{2})$

Now, the intermediate value theorem implies that for each $x \in (0,\pi)$ there exists γ s.t.



$A(\gamma) = \frac{x}{2}$, $2A(\gamma) = x$.

Furthermore, γ is unique because A is strictly decreasing.

Def. Given $x \in [0,\pi]$, define

$$\sin x = \sqrt{1-\cos^2 x}.$$

Theorem. If $x \in (0,\pi)$, then

1) $\cos'(x) = -\sin x$,

2) $\sin'(x) = \cos x$.

Proof. 1) Set $B = 2A$. Then

$B(\cos x) = x$, $x \in (0,\pi)$. Then

$B = \cos^{-1}$ and

$B' = 2A' = -\frac{1}{\sqrt{1-x^2}}$. Therefore,

$$\cos' x = \frac{1}{B'(\cancel{\cos} B^{-1}(x))} = \frac{1}{B'(\cos x)}$$

$$= \frac{1}{-\frac{1}{\sqrt{1-\cos^2 x}}} = -\frac{1}{\frac{1}{\sin x}} = -\sin x.$$

$\square$

2) Exercise.

We extend sin, cos to $[0, 2\pi]$ by

$\sin x = -\sin(2\pi - x)$, $x \in [\pi, 2\pi]$,

$\cos x = \cos(2\pi - x)$, — " —.

Extend cos, sin periodically to $\mathbb{R}$.

Other trig. functions (tan, cot, arcsin, ...) are defined standardly.

Defining log/exp.

Suppose $a > 0$. We know how to define a^r for $r \in \mathbb{Q}$. We also know that

$a^n \cdot a^m = a^{n+m}$, $m, n \in \mathbb{Q}$.

In other words, if $f(x) = a^x$, then

$$f(x) f(y) = f(x+y), \quad x, y \in \mathbb{Q}.$$

Question: can we find a function

$f\colon$ ~~[illegible]~~ $\mathbb{R} \to \mathbb{R}$ s.t. this formula holds for all $x, y \in \mathbb{R}$?
I look for f s.t. this equality holds <u>and</u> f is differeniable.
We could take $f = 0$ or $f = 1$.
So we look for such f, which also satisfies $f(1) = \alpha$, where $a \in \mathbb{R}$.

We have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x) \cdot f(h) - f(x)}{h}$$

$$= f(x) \lim_{h \to 0} \frac{(f(h) - 1)}{h}.$$

If the limit exists and equals α, then $\underline{f'(x) = \alpha f(x)}$. This implies f^{-1} satisfies

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\alpha f(f^{-1}(x))} = \frac{1}{\alpha x}.$$

Then we may have

$$f^{-1}(x) = \int_1^x \frac{1}{\alpha t}\,dt.$$

<u>Definition.</u> Define

$$\log x = \int_1^x \frac{1}{t}\,dt.$$

(for now, we assume $\alpha = 1$)

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Reminder def. $\log x = \int_1^x t^{-1}\,dt$.

Notation. $\log x = \ln x$.

~~$\log_a x = \log_a x$~~

Theorem. If $x, y > 0$, then

$$\log(xy) = \log x + \log y.$$

Proof. We know $\log'(x) = \frac{1}{x}$. Fix $y > 0$.

Set $g(x) = \log(xy)$. Then

$g'(x) = y \cdot \frac{1}{xy} = \frac{1}{x}$. This means

$g(x) = \log x + C$ for some constant C.

Take $x = 1$. Then $g(1) = \log 1 + C$,

which implies $g(1) = C$. But

$g(1) = \log(y \cdot 1) = \log y$. Thus,

$C = \log y$ and

$\log(xy) = g(x) = \log x + \log y$. $\square$

<u>Def.</u> The exponential function $\exp$ is defined as $\log^{-1}$.

<u>Remark.</u> $\log : (0,\infty) \to \mathbb{R}$ is continuous, strictly increasing.
$\exp : \mathbb{R} \longrightarrow (0,\infty)$ is contin. and ~~monotone~~ strictly increasing.

<u>Note.</u> If $n \in \mathbb{N}$, $\log x^n = n \log x$ and $\log \frac{x}{y} = \log x - \log y$.

<u>Theorem.</u> $\underline{\exp'(x) = \exp(x)}$ for all $x \in \mathbb{R}$.

<u>Proof.</u>

$$\exp'(x) = \left(\log^{-1}\right)'(x) = \frac{1}{\log'\left(\log^{-1}(x)\right)}$$

$$= \frac{1}{1/\log^{-1}(x)} = \log^{-1}(x) = \exp(x). \quad \square$$

<u>Theorem.</u> $\exp(x+y) = \exp(x) \cdot \exp(y)$.

<u>Proof.</u> Set $X = \exp(x)$ and $Y = \exp(y)$.
Then $x = \log X$, $y = \log Y$. We know

$\underline{x+y} = \log X + \log Y = \underline{\log(XY)}$.

Take exp of both sides:

$\underline{\exp(x+y)} = \exp(\log(XY))$

$= XY = \underline{\exp(x)\exp(y)}$. $\square$

For $a>0$, set $\underline{a^x = \exp(x \log a)}$

Then $\exp(x) = (\exp(1))^x$. We will denote $\exp(1)$ by e.

Easy to check: $(a^b)^c = a^{bc}$,

$a^1 = a$, $a^{b+c} = a^b \cdot a^c$, $a>0$, $b, c \in \mathbb{R}$.

What is $\exp(1)$?

<u>Note:</u> $\boxed{\lim_{y \to 0+} \frac{\log(1+y)}{y}}$

$= \underbrace{\lim_{y \to 0+} \frac{\log(1+y) - \log(1)}{y}} = \log'(1)$

$= 1$.

Now we use this equality with $y = \frac{1}{x}$ for some $x > 0$:

$$\lim_{x \to \infty} x \log\left(1 + \frac{1}{x}\right) = 1.$$

Now take exp of both sides

$$\exp\left(\lim_{x \to \infty} x \log\left(1 + \frac{1}{x}\right)\right) = \exp(1),$$

$$\lim_{x \to \infty}\left[\exp\left[x \log\left(1 + \frac{1}{x}\right)\right]\right] = \exp(1),$$

$$\lim_{x \to \infty}\left[\exp(1)^{x \log\left(1 + \frac{1}{x}\right)}\right] = \exp(1),$$

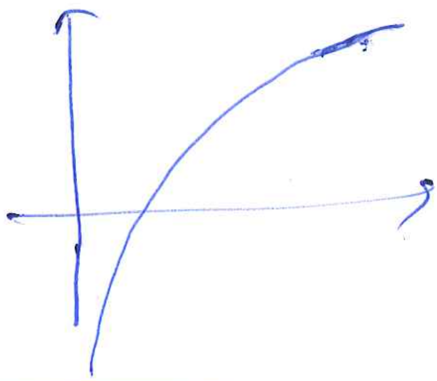
$$\lim_{x \to \infty}\left[\exp(1)^{\log\left(1 + \frac{1}{x}\right)}\right]^{x} = \exp(1).$$

$$\lim_{x \to \infty}\left(1 + \frac{1}{x}\right)^{x} = \exp(1).$$

Thus, $\exp(1) = e$, as defined earlier.

Note. The improper ~~ing~~tegral $\int_1^\infty x^{-1}\,dx$ diverges by the integral test.

Also, by the integral test,
$\sum_{n=1}^{\infty} n^p$ converges when $p < -1$.



Taylor series.

<u>Applications</u>: approximation of function, solving ODEs.

<u>Theorem</u> (Taylor's formula).
Assume f is differentiable $n+1$ times on some open interval containing a, b.

Then
$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \boxed{\frac{f^{(n)}(a)}{n!}(b-a)^n} \quad P_n(x)$$
$$R_n(x) \longrightarrow + \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1}$$
for some $\xi \in (a, b)$.

Note. $f(x) = P_n(x) + R_n(x)$, where $P_n(x)$ is called the nth Taylor polynomial and $R_n(x)$ is the remainder.

Note. $R_n(b) = \frac{1}{n!} \int_a^b (b-t)^n f^{(n+1)}(t)\, dt$

(cachy remainder).

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Assume that $R_n(x) \xrightarrow[n\to\infty]{} 0$ for some x. Then

$$f(x) = \lim_{n\to\infty} P_n(x) = f(a) + f'(a)(b-a)$$
$$+ \frac{f''(a)}{2!}(b-a)^2 + \ldots$$
$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

This is called the Taylor series for f. It is called a Maclaurin series if $a=0$.

__Intuition.__ Note that

$$f(a) = f(a) + \cancel{f'(a)(a-a)}$$

$$+ \cancel{\frac{f''(a)}{2!}(a-a)^2} + \cdots$$

$$\underline{f'(a)} = \underline{f'(a)\cdot 1} + \cancel{\frac{f''(a)}{2!}2(a-a)}$$

$$+ \cancel{\frac{f'''(a)}{3!}3(a-a)^2} + \cdots$$

Similarly, $f''(a) = \left(\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n\right)''\Big|_{x=a}$

Conclusion:

$$f^{(k)}(a) \text{ "equals" } \left(\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n\right)^{(k)}\Big|_{x=a}$$

__Facts.__ (here $a=0$)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

All these ~~f~~ series converge for all $x \in \mathbb{R}$.

<u>Example.</u> Let's compute the Taylor series for $f(x) = \log(x)$ about $a = 1$.

Need to know the derivatives of log at 1.

$\log(1) = 0$, $\log'(1) = \frac{1}{x}\Big|_{x=1} = 1$,

$\log''(1) = -\frac{1}{x^2}\Big|_{x=1} = -1$,

$\log'''(1) = 2 \cdot \frac{1}{x^3}\Big|_{x=1} = 2$,

$\log^{(4)}(1) = 2 \cdot (-3) \cdot \frac{1}{x^4}\Big|_{x=1} = -2 \cdot 3$,

Thus, $\log^{(n)}(1) = (-1)^{n-1} \cdot (n-1)!$.

Thus, $\log(x) = 0 + 1 \cdot (x-1) + \frac{(-1)}{2!}(x-1)^2$

$$+ \cdots + \frac{(-1)^{n-1}\cancel{(n-1)!}}{n\cancel{!}}(x-1)^n$$

$$+ \frac{\log^{(n+1)}(\xi)}{(n+1)!}(x-1)^{n+1}$$

$$= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}(x-1)^k$$

$$+ \frac{\log^{(n+1)}(\xi)}{(n+1)!}(x-1)^{n+1}.$$

Let's take $n = 3$ and $x = 1.1$.

$$\log(x) = 0 + 1(x-1) + \frac{(-1)}{2}(x-1)^2$$

$$+ \frac{1}{3}(x-1)^3 + \underbrace{\frac{\log^{(4)}(\xi)}{4!}(x-1)^4}_{R_3(1.1)}$$

for $\xi \in [1, x]$.

We take $x = 1.1$.

$$\log(1.1) = 0.1 - \frac{1}{2}0.1^2 + \frac{1}{3}\cdot 0.1^3$$

$$+ R_3(1.1).$$

$$= 0.095333\ldots + R_3(1.1).$$

We ~~kom~~ know

$$\cancel{\log}\,|R_3(1.1)| = \left|\frac{\log^{(4)}(\xi)}{4!}\,0.1^4\right|$$

$$= \left|\frac{-\cancel{2}\cdot\cancel{3}}{\cancel{2}\cdot\cancel{3}\cdot 4\,\xi^4}\cdot 0.1^4\right|$$

$$= \left|-\frac{1}{4\xi^4}\cdot 0.1^4\right| \leq \left|-\frac{1}{4\cdot 1^4}\,0.1^4\right|$$

$$= 0.000025$$

Also, $R_3(1.1) < 0$. Thus,

$$\log 1.1 \in [0.095333\ldots - 0.000025,\; 0.095333\ldots].$$

<u>Example.</u> Use Maclaurin series for e^x to compute e to 5 decimal places.

We know $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$

$$+ \frac{x^n}{n!} + \frac{(e^x)^{(n+1)}\big|_{x=\xi}}{(n+1)!}\, x^{n+1}.$$